Formal Introduction to Spectral Risk Measures

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Section 1: Definition and Examples of Risks Measures

Risk Measures: Definition

A **risk measure** ρ associates a **monetary amount** $\rho(X)$ with a **random financial outcome** X so that

$$\text{prefer } X \text{ to } Y \iff \rho(X) \leq \rho(Y)$$

- Risk measure $\rho(X)$ represents
  - The **assets required** to **credibly promise to pay** X or
  - A financial measure of the **pain** suffered by assuming X
- Insurance view: **bigger** $\rho(X)$ corresponds to **greater *risk***

Risk Measures: Warning

Generally accepted usage means

- A risk measure determines **assets** and **not capital**
- A risk measure is **not** a measure of **economic capital**
- The risk of a certain liability with present value of a is a even though the economic capital is zero

Two Favorite Risk Measures

Value at Risk (VaR)

- $\text{VaR}_p(X)$= percentile $= F^{-1}(p) =$ quantile $= q(p)$
  - p close to 1 worst for losses
  - **Advantages**: always finite
  - **Disadvantages**: dodgy with respect to diversification

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- $\text{VaR}_p(X)$ is the $N(1-p)$th largest observation from a sample of N simulated events

Two Favorite Risk Measures

Tail VaR (TVaR)

- $\text{TVaR}_p(X) =$ conditional average of the worst $1-p$ outcomes $= \dfrac{1}{1-p}\displaystyle\int_p^1 q(p)dp$
  - p close to 1 worst for losses
  - $p = 0$ corresponds to expected loss
  - $p = 1$ corresponds to least upper bound of losses
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  - **Disadvantages**: not always finite
- Also known as ES, AVaR, CVaR, CTE; there are some technical differences for non-continuous X

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Section 2: Distortion Function and Spectral Measures Recap

Distortion Functions Price Thin Layers

Simplifying idea

- Break pricing problem into sub-problems of pricing thin layers
- Add!

Distortion Functions Price Thin Layers

Simplifying idea

- Break pricing problem into sub-problems of pricing thin layers
- Add!

Why simplifying?

- Thin layers only have total losses, no partial losses
- **Risk** of thin layer completely described by **one number**, called
  - Exceedance probability (EP) S, or
  - Probability of attachment, or
  - Expected loss (EL)

Distortion Functions Price Thin Layers

Linking risk and price

- **Price** of thin layer therefore also described by **one number**, called
  - Rate-on-line (ROL), or
  - Risk adjusted or distorted probability, or
  - State-price

Distortion Functions Price Thin Layers

Linking risk and price

- **Price** of thin layer therefore also described by **one number**, called
  - Rate-on-line (ROL), or
  - Risk adjusted or distorted probability, or
  - State-price
- **Distortion function** g : thin layer risk $\mapsto$ price captures relationship between risk and price
  - g is a function $[0,1] \to [0,1]$
  - Risk averse implies $g(s) \geq s$ for all $s \in [0,1]$

Distortion Function to Risk Measure

- Associate a **risk measure** ρ_g to a distortion function g by analogy with $\mathsf{E}(X)$

$$\begin{aligned}\mathsf{E}(X) &= \int_0^\infty S(x)dx \\ &= \int_0^\infty xf(x)dx \\ &= \int_0^1 F^{-1}(p)dp\end{aligned}$$

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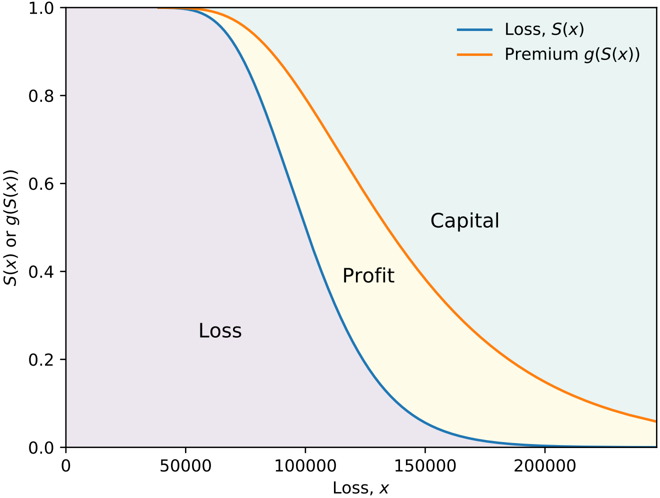
- Function g' on lower right measures care/care-more along the risk spectrum p, hence **spectral risk measure**
- $\mathsf{E}(X)$ corresponds to ρ_g with $g(s) = s$ the identity

Total Risk: Summing Over Thin Layers Using $g(S(x))$



Beware...

$$\boxed{\int_0^\infty x f(x) dx = \int_0^\infty S(x) dx}$$

But... generally for $0 < a < \infty$

$$\boxed{\int_0^a x f(x) dx \neq \int_0^a S(x) dx}$$

- The right hand side includes a full limit loss
- The left hand side does not

Section 3: Properties of Risks Measures

Properties of Risk Measures

Property	Meaning
Translation invariance	adding cash exactly lowers risk
Monotone	more loss $\implies$ more risk
Positive homogeneous	scale irrelevant; subtle and insidious
Sub-additive	mergers do not increase risk
Law invariant	risk only depends on loss, not cause
Comonotonic additive	no diversification $\implies$ no diversification credit

Properties of Risk Measures

- **Coherent** means all of
  - Translation invariant
  - Monotone
  - Positive homogeneous
  - Sub-additive

Properties of Risk Measures

- **Coherent** means all of
  - Translation invariant
  - Monotone
  - Positive homogeneous
  - Sub-additive
- Convex
  - Coherent without positive homogeneity (better)
  - Risk of weighted average $\leq$ weighted average of risk

See Appendix A for details

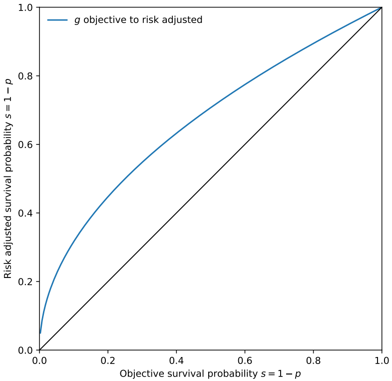
Properties of Distortion Functions

Correspondence between properties of ρ_g and g

Property of ρ_g	**Property of g**
Translation invariance	$g(0) = 0$, $g(1) = 1$
Monotone	g is increasing
Positive homogeneous	True for all g
Sub-additive	g is concave
Law invariant	True for all g
Comonotonic additive	True for all g

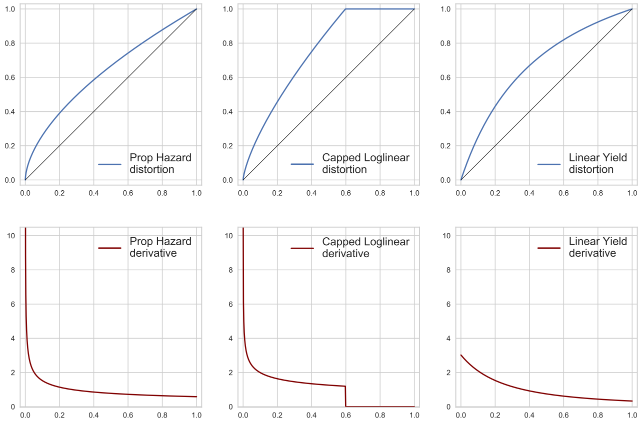
Correspondence between properties of ρ_g and g

- g is concave (blue)
  - bows up above diagonal
  - g' decreasing
  - **care-less** about **smaller**, higher probability, losses
- $g'(1-p)$ is the **state price density** of the $p=F(x)$ percentile loss, $\rho_g(X)=\int x g'(S(x))f(x)dx$
- g maps EL to ROL, objective probability to risk adjusted probability, "P to Q"

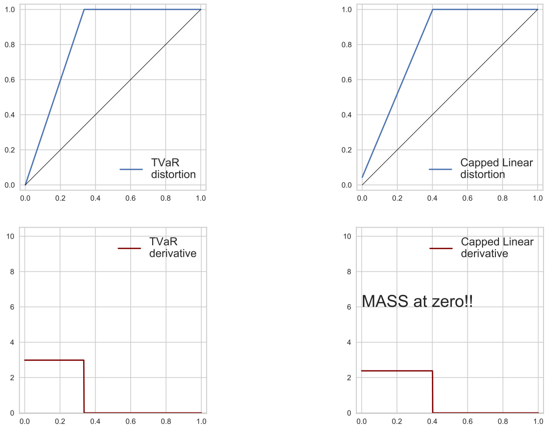


Section 4: Example Distortion Functions

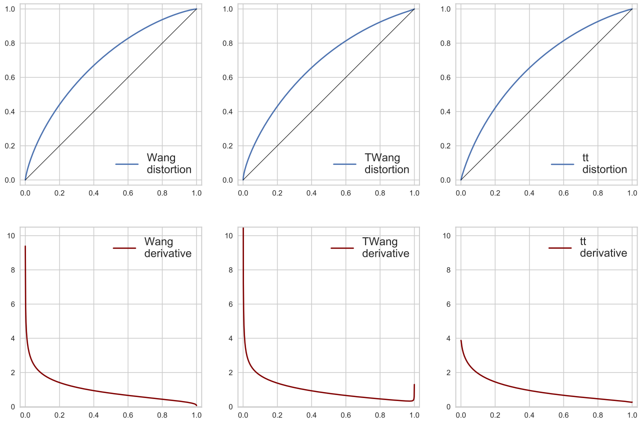
Proportional Hazard, Capped Loglinear and Linear Yield



TVaR and Capped Linear

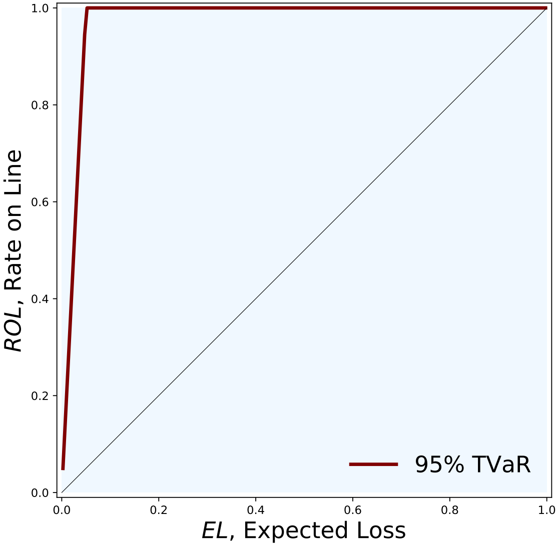


Wang, t-Wang, and tt



Section 5: TVaR Example

Distortion Function Behind TVaR



TVaR-Person View of the World

- Only events in the top $1 - p$ percentile can occur
- TVaR-Person regards events smaller than VaR_p as **impossible**
- $g'(x) = 0$ for these smaller losses
- **Big Honking Problem**: Does TVaR-Person give away coverage below VaR_p?

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- **Big Honking Problem**: Does TVaR-Person give away coverage below VaR_p?
- VaR-person regards all events as equal to VaR_p
- That **is** a BHP: free cover above VaR_p

TVaR-Person View of the World

Does TVaR-Person give away coverage below VaR_p?

- **Partial-loss only** cover, paying only for losses in layer: **free**
  - TVaR-person's simulations do not include any partial losses
  - They do not believe partial losses possible
  - They would not understand demand for product
  - Partial-loss only covers are not actually sold

TVaR-Person View of the World

Does TVaR-Person give away coverage below VaR_p?

- **Partial-loss only** cover, paying only for losses in layer: **free**
  - TVaR-person's simulations do not include any partial losses
  - They do not believe partial losses possible
  - They would not understand demand for product
  - Partial-loss only covers are not actually sold
- For **traditional layer**, paying full limit losses for over-the-top: price $>$ expected loss
  - All TVaR-person's simulated losses $\geq \text{VaR}_p$
  - **All simulated losses are full limit losses**
  - TVaR-person premium $yS(a) = y \times 1 = y =$ limit $>$ expected loss to layer

Appendix A: Properties of Risk Measures

Properties of Risk Measures

Translation Invariance (**TI**)

- Lowering a loss by a fixed, certain amount a lowers risk by the same amount: $\rho(X - a) = \rho(X) - a$
- Requires
  - ρ denominated in dollars, so $\rho(X) - a$ makes sense
- Examples
  - Expected value
  - VaR, TVaR
  - Scenario loss
- Rules out
  - Standard deviation, variance, $\mathsf{Var}(X + a) = \mathsf{Var}(X)$
  - All higher central moments
  - EPD
  - Probability of downgrade
  - Capital adequacy *ratio*

Properties of Risk Measures

Monotonicity (**MON**)

- The more I owe the worse it is: if $X \geq Y$ for all outcomes then $\rho(X) \geq \rho(Y)$
- Equivalently, if $X \geq 0$ for all outcomes then $\rho(X) \geq 0$
- Examples
  - Expected value
  - VaR, TVaR
- Rules out
  - Standard deviation, e.g. uniform$(0, 1) < 1$ but sd(uniform) > 0
  - Other central moments

Properties of Risk Measures

Positive Homogeneity or Scaling (**PH**)

- Scales, $\rho(\lambda X) = \lambda \rho(X)$ for all $\lambda > 0$
- Highly dodgy: ask LTCM; unclear meaning of λX
- Examples
  - VaR
  - SD
  - Scenario loss
- Rules out
  - Variance

Properties of Risk Measures

Sub-Additivity (**SA**)

- Respect diversification: $\rho(X + Y) \leq \rho(X) + \rho(Y)$
- Not without controversy, regulators find too much diversification
- Examples
  - Expected value
  - TVaR
- Rules out
  - VaR (because of thick tails or weird dependency structure)
  - Variance

Properties of Risk Measures

Coherent (**COH**)

- Translation invariance, monotonic, positive homogeneous, sub-additive together called a **coherent** risk measure
- (**TI**, **MON**, **PH**, **SA**) $\iff$ (**COH**)
- Examples
  - TVaR
  - Average of TVaRs at different thresholds
  - Worst of specified set of scenarios (Lloyds RDS)
- Rules out
  - Variance
  - VaR

Properties of Risk Measures

Law Invariant (**LI**)

- If loss outcome contains all relevant information to determine risk then ρ is called law invariant (**LI**)
- LI means the risk only depends on the distribution F of X
  - Makes sense: an entity's risk of insolvency only depends on its distribution of future change in surplus—the cause of loss is irrelevant to solvency
  - May not make sense: a dollar of loss from Florida hurricane is more expensive to transfer than a dollar from non-cat auto liability
  - Suitability depends on application
- LI is shorthand way to tailor events to entity's actual losses rather than common objective events
  - What is relevant to me?
- Examples
  - VaR, TVaR, SD
- Rules out
  - Scenarios

Properties of Risk Measures

Comonotonic Additive (**CA**)

- Two random variables X and Y are **comonotonic** if either
  - $(X(\omega_1) - Y(\omega_1))(X(\omega_2) - Y(\omega_2)) \geq 0$ for all $\omega_1, \omega_2 \in \Omega$, i.e. samples from (X, Y) lie on an upward sloping line, or equivalently
  - $X = h(Z)$ and $Y = h(Z)$ for an increasing function h and third random variable Z
- If X and Y have quantile functions q_X and q_Y and given a uniform variable U, $(q_X(U), q_Y(U))$ is a comonotonic bivariate distribution with marginals X and Y
- If X and Y are comonotonic then $q_{X+Y} = q_X + q_Y$
- A risk measure is **comonotonic additive** (**CA**) if $\rho(X + Y) = \rho(X) + \rho(Y)$ whenever X, Y are comonotonic
- CA implies PH: $\rho(2X) = \rho(X + X) = \rho(X) + \rho(X) = 2\rho(X)$ etc.
- Examples
  - VaR, TVaR
- Rules out
  - Scenarios

Appendix B: Distortion Function and Integrals

Details of Distortion Functions

Table 3: Parameters and Definitions of Distortion Functions

Distortion	Parameters	$g'(0)$	Formula
Proportional Hazard	$b \leq 1$	Unbounded	$g(s) = s^b$
Capped Log-linear	$a, b,\ 0 < b \leq 1$	Unbounded	$g(s) = \min(1, \exp(a + b\log(s)))$
Linear Yield	r_o, r_K	Bounded	$g(s) = \dfrac{r_o + s(1 + r_K)}{1 + r_o + r_K s}$
TVaR	$\alpha \in [0, 1]$	Bounded	$g(s) = \min(1, s/(1 - \alpha))$
Capped Linear	a, b	Mass	$g(s) = \min(1, a + bs)$
Wang	λ	Unbounded	$g(s) = \Phi(\Phi^{-1}(s) + \lambda)$
t-Wang	λ, df	Invalid	$g(s) = t_{df}(\Phi^{-1}(s) + \lambda)$
t-t	λ, df	Unbounded	$g(s) = t_{df}(t_{df}^{-1}(s) + \lambda)$

- Images show t-Wang distortion is not concave and hence does not define a coherent risk measure
- Capped linear a.k.a. linear